

Non-Equilibrium Relaxing Gas Flow Behind Three Dimensional Unsteady Curved Shock Wave

V. D. Sharma* and Radhe Shyam

Applied Mathematics Section, Institute of Technology, Banaras Hindu University, Varanasi 221005, India

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A shock wave is assumed to exist in a three-dimensional unsteady flow of a relaxing gas. The variation of flow parameters at any point behind the shock surface is determined in terms of the shock geometry and the upstream flow conditions. The expressions for the vorticity and the curvature of a streak line at the rear of the shock surface are also determined in terms of the known quantities.

1. Introduction

Problems of the flow of non-equilibrium gases have become increasingly important because of the elevated temperatures encountered in high speed flows, and much effort has accordingly been devoted to re-examining gas flows in the light of chemical non-equilibrium. However, much less interest has been shown in the geometry of such fluid flows. The problems dealing with the kinematical and geometrical properties of fluid flows in classical gasdynamics, magnetogasdynamics and relativistic gasdynamics have drawn attention of a number of researchers [1–4]. The main object of this paper is to study how non-equilibrium relaxation effects modify the results of classical gasdynamics. The existence of a shock wave is assumed and the flow at the rear of the shock surface is examined. A coordinate transformation is used which offers a convenient way of treating this problem. The present method can, however, be employed to vibrational excitation, ionization, dissociation etc.

The basic equations are [5] (p. 37)

$$\begin{aligned} \partial p / \partial t + u_i p_{,i} + \varrho c_f^2 u_{i,i} \\ = - c_f^2 ((A/T) \varrho_s + \varrho q) \omega(p, s, q), \end{aligned} \quad (1.1)$$

$$\varrho \partial u_i / \partial t + \varrho u_j u_{i,j} + p_{,i} = 0, \quad (1.2)$$

$$T(\partial s / \partial t + u_i s_{,i}) = A \omega(p, s, q), \quad (1.3)$$

$$\partial q / \partial t + u_i q_{,i} = \omega(p, s, q), \quad (1.4)$$

* Present address: Department of Aerospace Engineering, University of Maryland, College Park, Maryland 20742, U.S.A.

Reprint requests to Dr. V. D. Sharma, Applied Mathematics Section, Institute of Technology, Banaras Hindu University, Varanasi 221005/Indien.

where the summation convention on repeated indices is employed, and $\partial/\partial t$ and a comma followed by an index (say i) denote partial differentiation with respect to time and the corresponding coordinate, x_i , respectively. The range of Latin indices is taken to be 1, 2, 3 and the symbols appearing in (1.1) to (1.4) are as follows: u_i are the gas velocity components, p is the pressure, s the entropy, q the progress variable characterizing the extent of internal transformation in the gas, ω the rate of internal transformation which is assumed to be a known function of p , s and q , and c_f is the frozen sound speed given by $c_f^2 = (\partial p / \partial \varrho)_{s,q}$. The symbols ϱ , T and A , which denote, respectively, the density, temperature, and the affinity of internal transformation characterized by the variable q , are regarded as functions of p , s and q , and are given by

$$\varrho^{-1} = h_p, \quad T = h_s \quad \text{and} \quad A = -h_q,$$

where h is the specific enthalpy related to p , s and q through the canonical equation of state $h = h(p, s, q)$, and the letter subscripts p , s or q denote partial derivatives with respect to the indicated variable while holding the remaining variables in the set (p, s, q) fixed.

2. Shock Configuration and Jump Relations

Let the shock configuration in three-dimensional unsteady flows be represented by continuously differentiable functions $\bar{x}_i = \bar{x}_i(y^\alpha, t)$ (in this and in what follows the range of Latin indices is 1, 2, 3 and that of Greek indices is I, II; a repeated index implies summation unless stated otherwise) with respect to a frame of reference in which it is at rest, where x_i are the Cartesian coordinates of a point on

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locity defined as the velocity of a point of the shock surface S_0 in a direction normal to the surface. A point with fixed coordinates y^α on the shock surface S_0 at every instant moves with the shock velocity $G\tilde{n}_i$, where \tilde{n}_i are the components of the unit normal to the shock surface S_0 at this point. Let $\delta/\delta t$ denote the time derivative as apparent to an observer moving with the velocity $G\tilde{n}_i$, then the $\delta/\delta t$ derivative of a quantity f is defined as in [6]:

$$\delta f/\delta t = \partial f/\partial t + G\tilde{n}_i f_{,i}. \quad (2.1)$$

Thus, (1.1), (1.2), (1.3) and (1.4) may be put into the forms

$$\delta p/\delta t + (u_i - G n_i) p_{,i} + \varrho c_{\text{r}}^2 u_{i,i} + A = 0, \quad (2.2)$$

$$\varrho \delta u_i/\delta t + \varrho (u_j - G n_j) u_{i,j} + p_{,i} = 0, \quad (2.3)$$

$$\delta s/\delta t + (u_i - G n_i) s_{,i} = (A/T) \omega(p, s, q), \quad (2.4)$$

$$\delta q/\delta t + (u_i - G n_i) q_{,i} = \omega(p, s, q), \quad (2.5)$$

where

$$A = c_{\text{r}}^2 ((A/T) \varrho_s + \varrho_q) \omega(p, s, q).$$

Let a quantity f , if evaluated upstream (down stream) from the shock be denoted by $f_1(f)$. Let $[f]$ denote the jump in the quantity enclosed as it crosses the shock surface S_0 (a horizontal bar above a quantity denotes its value just at the shock surface S_0). Then the jump conditions expressing the values of flow variables just behind in terms of those just ahead of the shock surface S_0 are [5] (p. 40)

$$[\bar{u}_i] = -\bar{\zeta}(1 + \bar{\zeta})^{-1} \bar{V}_{1n} \bar{n}_i, \quad (2.6)$$

$$[\bar{p}] = \bar{\zeta}(1 + \bar{\zeta})^{-1} \bar{\varrho}_1 \bar{V}_{1n}^2, \quad (2.7)$$

$$[\bar{q}] = 0, \quad (2.8)$$

$$[\bar{h}] = \bar{\zeta}(2 + \bar{\zeta}) \bar{V}_{1n}^2/2(1 + \bar{\zeta})^2, \quad (2.9)$$

where $\bar{V}_i = \bar{u}_i - G\bar{n}_i$ and $\bar{V}_{1n} = \bar{V}_{1i}\bar{n}_i$. $\bar{\zeta} = [\bar{\varrho}]/\bar{\varrho}_1$, in view of (2.9), is a function of p , s , and q , and is defined as the density strength of the shock. Since, from the jump conditions [5] (p. 40 and 41) p , s , and q just at the rear of the shock are known in terms of the state one variables, it follows that $\bar{\zeta}$ is a known function of \bar{p}_1 , \bar{s}_1 and \bar{q}_1 .

Let x^i be the coordinates of a fluid particle at a point P in the region behind the shock surface S_0 at a distance ξ measured along its streak-line from the shock front. Let us consider a surface S through P, which is such that when $\xi \rightarrow 0$, S coincides with

the shock surface, say S_0 , and y^α are its Gaussian coordinates. Let G be the instantaneous shock velocity of the shock surface S_0 . Then the configuration of such a surface S is given by

$$x_i = x_i(y^{\text{I}}, y^{\text{II}}, \xi), \quad \partial x_i/\partial \xi = V_i/v \quad (2.10)$$

with the initial conditions

$$x_i(y^{\text{I}}, y^{\text{II}}, 0) = \bar{x}_i(y^{\text{I}}, y^{\text{II}}), \quad (\partial x_i/\partial \xi)_{\xi=0} = \bar{V}_i/\bar{v},$$

where $v^2 = V_i V_i$, and a horizontal bar above a quantity denotes its value on the shock surface S_0 . In view of (2.10), we have

$$f_{,j} = \frac{\partial f}{\partial \xi} \xi_{,j} + y_{,j}^\alpha f_{,\alpha}, \quad (2.11)$$

$$x_{i,j} = \delta_{ij} = \frac{V_i}{v} \xi_{,j} + x_{i,\alpha} y_{,j}^\alpha \quad (2.12)$$

with δ_{ij} being the Kronecker delta.

Multiplying (2.12) by n_i and $\varepsilon^{\beta\gamma} \varepsilon_{ilk} V_l x_{k,\gamma}$, we get

$$\xi_{,j} = (v/V_n) n_j; \quad y_{,j}^\alpha = \varepsilon^{\alpha\beta} \varepsilon_{ijk} V_i x_{k,\beta}/V_n; \quad V_j y_{,j}^\alpha = 0, \quad (2.13)$$

where $V_n = V_i n_i$ and $\varepsilon^{\alpha\beta}$ and ε_{ijk} are permutation tensors of the surface and space, respectively.

3. Gradients of Flow Parameters Behind the Shock

In the region behind the shock surface, (2.2), (2.3), (2.4) and (2.5) transform to

$$\frac{\delta p}{\delta t} + v \frac{\partial p}{\partial \xi} + \varrho c_{\text{r}}^2 \left(\frac{v}{V_n} n_i \frac{\partial u_i}{\partial \xi} n_i + u_{i,\alpha} y_{,i}^\alpha \right) + A = 0, \quad (3.1)$$

$$\varrho \frac{\delta u_i}{\delta t} + \varrho v \frac{\partial u_i}{\partial \xi} + \frac{v n_i}{V_n} \frac{\partial p}{\partial \xi} + p_{,\alpha} y_{,i}^\alpha = 0, \quad (3.2)$$

$$\frac{\delta s}{\delta t} + v \frac{\partial s}{\partial \xi} = (A/T) \omega(p, s, q), \quad (3.3)$$

$$\frac{\delta q}{\delta t} + v \frac{\partial q}{\partial \xi} = \omega(p, s, q), \quad (3.4)$$

where use has been made of (2.11) and (2.13).

From (3.1)–(3.4), we obtain

$$v \frac{\partial u_i}{\partial \xi} = \frac{V_n^2}{\varrho (V_n^2 - c_{\text{r}}^2)} \cdot \left\{ \frac{n_i}{V_n} \left(\frac{\delta p}{\delta t} + \varrho c_{\text{r}}^2 u_{i,\alpha} + A \right) - \varrho \frac{\delta u_i}{\delta t} - p_{,\alpha} y_{,i}^\alpha \right\}, \quad (3.5)$$

$$v \frac{\partial p}{\partial \xi} = \frac{V_n^2}{(V_n^2 - c_t^2)} \cdot \left\{ \frac{c_t^2}{V_n} \left(\varrho \frac{\delta u_i}{\delta t} + p, \alpha y_{,i}^\alpha \right) n_i - \frac{\delta p}{\delta t} - \varrho c_t^2 u_i, \alpha y_{,i}^\alpha - A \right\}, \quad (3.6)$$

$$v \frac{\partial s}{\partial \xi} = - \frac{\delta s}{\delta t} + (A/T) \omega(p, s, q), \quad (3.7)$$

$$v \frac{\partial q}{\partial \xi} = - \frac{\delta q}{\delta t} + \omega(p, s, q). \quad (3.8)$$

Hence, by virtue of (2.11), we find that the gradients of the flow variables at any point behind the shock surface are given by the equations

$$u_{i,j} = \frac{v}{V_n} \frac{\partial u_i}{\partial \xi} n_j + u_i, \alpha y_{,j}^\alpha, \quad (3.9)$$

$$p_{,j} = \frac{v}{V_n} \frac{\partial p}{\partial \xi} n_j + p, \alpha y_{,j}^\alpha, \quad (3.10)$$

$$s_{,j} = \frac{v}{V_n} \frac{\partial s}{\partial \xi} n_j + s, \alpha y_{,j}^\alpha, \quad (3.11)$$

$$q_{,j} = \frac{v}{V_n} \frac{\partial q}{\partial \xi} n_j + q, \alpha y_{,j}^\alpha, \quad (3.12)$$

where $y_{,j}^\alpha$, $v \frac{\partial u_i}{\partial \xi}$, $v \frac{\partial p}{\partial \xi}$, $v \frac{\partial s}{\partial \xi}$ and $v \frac{\partial q}{\partial \xi}$ are given by (2.13), (3.5), (3.6), (3.7) and (3.8), respectively.

Under the initial conditions of the transformation (2.10) as $\xi \rightarrow 0$, the quantities u_i , V_i , p , s , q , n_i and c_t tend to \bar{u}_i , \bar{V}_i , \bar{p} , \bar{s} , \bar{q} , \bar{n}_i and \bar{c}_t , respectively. Accordingly, we conclude that the gradients of flow and field parameters at any point in a non-equilibrium relaxing gas flow just behind an oblique shock are determined by the Eqs. (3.9) to (3.12) if the horizontal bar is placed above every quantity involved there and in (2.13) and (3.5) to (3.8), respectively.

It is obvious from the expressions for $v \partial u_i / \partial \xi$, $v \partial p / \partial \xi$, $v \partial s / \partial \xi$ and $v \partial q / \partial \xi$ that the gradients of flow parameters behind the shock are uniquely determined if, and only if,

$$V_n \neq 0; \quad V_n^2 - c_t^2 \neq 0.$$

The above conditions suggest that the discontinuity appearing in the flow under consideration can neither be a tangential nor a sonic one.

It is to be noted here that the energy equation and the equation of state are essentially needed for the calculation of flow gradients.

4. Determination of Unknown Quantities

In the subsequent discussion, we shall omit the horizontal bar above each quantity for the sake of simplicity. Since the flow quantities on the upstream side of the shock are assumed to be uniform and known, the values of the flow parameters just behind the shock are explicitly given in terms of known quantities by (2.6) to (2.8). Thus, the quantities expressed as functions of V_{1n} , p_1 , q_1 , n_i , G , c_t , ζ , the metric tensor of the shock surface and their derivatives along the shock surface will allow for determination. Such quantities depend only upon the shape, speed and strength of the shock and the conditions imposed on the upstream side of the shock. Thus, all the unknown quantities involved in the expressions for gradients of flow parameters behind the unsteady shock wave allow for determination. In the subsequent discussion we shall explicitly calculate the unknown quantities in terms of known quantities in the case of a non-equilibrium flow of a relaxing gas.

The time derivative of n_i as apparent to an observer moving with the shock is given by [7]

$$\delta n_i / \delta t = - \alpha^\alpha \beta G, \alpha x_{i,\beta}, \quad (4.1)$$

where $\alpha^\alpha \beta$ are the contravariant components of the metric tensor of the shock surface.

We assume that the flow upstream from the shock is uniform and known and the lines of curvature of the shock surface are Gaussian coordinate curves. Keeping in mind this assumption and making use of (4.1), we obtain

$$\delta_t V_{1n} = - (u_1^\alpha G, \alpha + \delta_t G). \quad (4.2)$$

Now, applying the operator $\delta_t (= \delta / \delta t)$ on (2.6) to (2.8) and using (4.1) and (4.2), we get the δ_t derivatives involved in the expressions for gradients in terms of known quantities depending upon the shape, speed and strength of the shock and the flow parameters just in front of the shock in the following form:

$$\begin{aligned} \frac{\delta u_i}{\delta t} &= \frac{\zeta}{1 + \zeta} \{ V_{1n} \alpha^\alpha \beta G, \alpha x_{i,\beta} \\ &\quad + n_i (u_1^\alpha G, \alpha + \delta_t G) \} - \frac{n_i V_{1n}}{(1 + \zeta)^2} \delta_t \zeta, \end{aligned} \quad (4.3)$$

$$\frac{\delta p}{\delta t} = -\frac{\varrho_1 V_{1n}}{1+\zeta} \left\{ 2\zeta(u_{1\alpha} G_{,\alpha} + \delta_t G) - \frac{V_{1n}}{1+\zeta} \delta_t \zeta \right\}, \quad (4.4)$$

$$\frac{\delta q}{\delta t} = 0. \quad (4.5)$$

Using Weingarten's formula for the surface derivative of n_i as given by

$$n_{i,\alpha} = -a^{\beta\gamma} b_{\gamma\alpha} x_{i,\beta}, \quad (4.6)$$

where $b_{\gamma\alpha}$ are the covariant components of the second fundamental tensor of the surface, we obtain

$$V_{1n,\alpha} = -(K_\alpha u_{1\alpha} + G_{,\alpha}) \quad (\alpha \text{ unsummed}), \quad (4.7)$$

where K_α are the normal curvatures of the shock surface given by

$$K_\alpha = b_{\alpha\alpha}/a_{\alpha\alpha} \quad (\alpha \text{ unsummed}). \quad (4.8)$$

As the coordinate curves on the shock surface are lines of curvature, we have $a_{12} = b_{12} = 0$. Now, differentiating (2.6) to (2.8) with respect to y^α and making use of (4.6) and (4.7), we get expressions for the quantities $u_{i,\alpha}$, $p_{,\alpha}$ and $q_{,\alpha}$ involved in the expressions for the gradients in terms of known quantities depending upon the shape, speed and strength of the shock and the flow parameters upstream from the shock in the form

$$u_{i,\alpha} = \frac{\zeta}{1+\zeta} \{ V_{1n} a^{\beta\gamma} b_{\gamma\alpha} x_{i,\beta} + n_i (K_\alpha u_{1\alpha} + G_{,\alpha}) \} - \frac{n_i V_{1n}}{(1+\zeta)^2} \zeta_{,\alpha}, \quad (4.9)$$

$$p_{,\alpha} = -\frac{\varrho_1 V_{1n}}{1+\zeta} \left\{ 2\zeta (K_\alpha u_{1\alpha} + G_{,\alpha}) - \frac{V_{1n}}{1+\zeta} \zeta_{,\alpha} \right\}, \quad (4.10)$$

$$q_{,\alpha} = 0. \quad (4.11)$$

5. Entropy Gradient

Using the (2.7), (2.8), (2.9), (3.7), (4.4), (4.5), (4.10), (4.11) and $h = h(p, s, q)$, we get the quantities $\partial s/\partial \xi$ and $s_{,\alpha}$ in the form

$$\frac{\partial s}{\partial \xi} = \frac{1}{vT} \left\{ \frac{V_{1n} \zeta}{(1+\zeta)^2} (u_{1\alpha} G_{,\alpha} + \delta_t G) + A \omega(p, s, q) \right\}, \quad (5.1)$$

$$s_{,\alpha} = -\frac{V_{1n} \zeta^2}{T(1+\zeta)^2} (K_\alpha u_{1\alpha} + G_{,\alpha}). \quad (5.2)$$

Equations (5.1) and (5.2) determine, respectively, the variation of entropy along the streak-line at the shock and its variation along the shock surface in terms of known quantities. Consequently, the entropy gradient at any point just at the rear of an unsteady shock in a non-equilibrium relaxing gas flow is known from (3.11).

The third term on the right hand side of (5.1) is the contribution of non-equilibrium relaxation present in the flow, which will disappear if the flow is in equilibrium so that $\omega(p, s, q) = 0$.

The variation of entropy behind the shock leads to the presence of vorticity which can be determined as follows: The components of vorticity generated behind the shock are given by

$$\omega_i = \varepsilon_{ijk} u_{k,j}. \quad (5.3)$$

Substituting for $u_{k,j}$ in (5.3), we obtain

$$\begin{aligned} \omega_i = & -\zeta a^{\alpha\beta} \varepsilon_{ijk} n_j x_{k,\beta} G_{,\alpha} \\ & + \zeta V_n a^{\beta\gamma} b_{\gamma\alpha} \varepsilon_{kij} x_{k,\beta} y_{,j}^\alpha \\ & + \zeta (1+\zeta)^{-1} (K_\alpha u_{1\alpha} + G_{,\alpha}) \varepsilon_{ijk} n_j y_{,k}^\alpha. \end{aligned} \quad (5.4)$$

Now, making use of (2.13), (4.8) and the geometrical relation

$$n_i = \frac{1}{2} \varepsilon^{\alpha\beta} \varepsilon_{ijk} x_{j,\alpha} x_{k,\beta}$$

in (5.4), we obtain the expression for the vorticity generated by the shock in the form

$$\omega_i = -\zeta^2 (1+\zeta)^{-1} (K_\alpha u_{1\alpha} + G_{,\alpha}) \varepsilon^{\alpha\beta} x_{i,\beta}, \quad (5.5)$$

which shows that the magnitude of the vorticity generated by a shock of given speed, shape and strength depends only on the tangential component of the velocity and is independent of the form of the equation of state. Further, it shows that the vorticity generated by an oblique shock is strongly dependent upon ζ and becomes very large as ζ becomes large for a given shape and speed of the shock. Also, it follows from (5.5) that the normal component of the vorticity is always zero. It is interesting to note that for a plane flow the expression (5.5) reduces to

$$\omega_i = -\zeta^2 (1+\zeta)^{-1} k u_{1t}, \quad (5.6)$$

where k is the curvature of the shock curve and u_{1t} is the component of the fluid velocity tangential to the shock curve.

The relation (5.6) is exactly the same as that derived in [8] (p. 36).

6. Curvature of a Streak-line

If v_i are the components of the unit vector along the principal normal of the streak-line, then we have

$$k v_i = \frac{\partial}{\partial \xi} \left(\frac{u_i}{u} \right) = u_j u_{i,j} / u^2 - u_i u_j u_k u_{k,j} / u^4, \quad (6.1)$$

where $u^2 = u_i u_i$. Multiplying (6.1) by v_i and summing, we obtain

$$k = v_i u_j u_{i,j} / u^2,$$

which, on using (6.1), yields

$$k^2 = (1/u^4) \{ (u_j u_{i,j})^2 - (u_i u_j u_{i,j} / u)^2 \}. \quad (6.2)$$

Substituting the value of $u_{i,j}$ from (3.9) in (6.2), we find that the curvature of the streak-line at the rear of the shock is fully determined in terms of known quantities.

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